Labor's Shares – Aggregate and Industry: Accounting for Both in a Model of Development with Induced Innovation[†]

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Abstract**:** The relative stability of aggregate labor's share constitutes one of the great macroeconomic ratios. However, changes in individual industry labor's shares are essentially statistically independent of one another, and the average values of industry labor's shares widely vary. We present a two-sector model of economic development with induced innovation that can rationalize these phenomena as well as several other empirical regularities of real economies. Specifically, the model can account for (i) manufacturing industries becoming increasingly capital-intensive over time despite (ii) an increase in the relative price and share in total output of service industries; and (iii) aggregate labor's share remains within a narrow range despite (iv) individual industry labor's shares being uncorrelated with one another over time. In the long-run the model economy can attain either a neoclassical steady-state or endogenous growth, giving it the potential to account for a wide range of real world development experiences.

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I. Introduction

The relative stability of aggregate labor's share constitutes one of the great macroeconomic ratios. Relative stability usually connotes a horizontal trend *and/or* a narrow range within which labor's share remains over time. For example, in the US from 1958 to 1996 labor's share of aggregate value-added remained between 65 and 70 percent (**Figure 1**, Young (2005)).

However, Solow (1958) and Young (2005) have demonstrated that changes in individual US industry labor's shares are essentially statistically independent of one another. Furthermore, the average values of industry labor's shares range from under 30 percent to above 80 percent. **Table 1** recreates Young's (2005) decomposition of US aggregate labor's share changes, aggregated from data on 35 industries, into "within-industry," "between-industry," and "covariance" components.[1] The first component represents changes attributable to changes in industry labor's shares while the later two represent, respectively, those attributable to changes in value-added shares and positive or negative comovements between labor's shares and value-added shares. Volatility of aggregate US labor's share is almost identically accounted for by the within-industry component.

The above point is made visually in **Figures 1a** & **1b** which plot 1958 through 1996 labor's shares and shares in aggregate value-added for major US industry groups: agriculture, manufacturing and services. (See also the summary statistics in **Table 2.**) First, average labor's shares are similar even though their volatilities and trends over time vary considerably. Second, there is no suggestion of strong positive or negative

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[1] The decomposition is borrowed from Foster et al (2001) who initially apply it to US aggregate productivity growth. Garrido Ruiz (2005), in turn, adopted the decomposition to analyze labor's share in the Spanish economy.

comovement. (Even the downward-trending agriculture share and upward-trending services share represent only a -0.601 correlation.) Third, the trends in value-added shares suggest an increase in aggregate labor's share – labor's and value-added shares for services both increase while for manufacturing they decrease – while, from **Figure 1**, it is clear that such an increase does not materialize.

Indeed, **Table 3** presents industry changes in labor's and value-added shares along with each industry's cumulative contribution to the change in aggregate labor's share from 1958 through 1996. For every industry, save one, the cumulative contribution to the change in aggregate labor's share is modest. The exception is the services industry which, *ceteris paribus*, contributes a 10 percent increase to aggregate labor's share. Without the services industry, quite possibly aggregate labor's share would have decreased notably over the time period.[2]

Beyond the US, Garrido Ruiz (2005), applying the same decomposition as Young (2005) to Spanish data, also finds a dominant role for the within-industry component. More broadly, aggregate labor's shares do not seem to increase or decrease significantly over time in either industrialized or in developing economies. Gollin (2002), in a sample of 31 countries at various stages of development, calculates labor's shares all in the range of 65 to 80 percent. Similar to the US case, cross-country differences in industry value-added shares account for little of the cross-country differences in labor's shares.

We present a theory of economic development and growth that accounts for labor's share behavior at both the aggregate and the industry levels. A model, based on Zuleta (2005), is constructed with two sectors – one with a labor-only technology and one

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[2] That aggregate labor's share would tend to decrease over time in the absence of the service industry's contribution is a feature of the model presented below.

where both capital and labor are used and labor-saving innovations can be pursued.[3] This model features (i) one sector becoming increasingly capital-intensive over time, (ii) an increase in the relative price and share in total output of goods from the labor-only sector, (iii) stable aggregate labor's share, and (iv) sectoral labor's shares that are uncorrelated over time. Additionally, the model (v) supports either a neoclassical steady-state or long-run, endogenous growth.

We interpret the labor-only and capital-using sectors and, respectively, the services and manufacturing (or goods) industries. The basic storyline underlying the model is increasing labor productivity in manufacturing due to labor-saving innovations and, concurrently, an increasing share of total labor supply employed in the services industries. If endogenous growth is achieved, this uneven productivity and labor supply growth across industries leads towards a zero manufacturing labor's share and a services labor's share of unity. (This ever-greater share of labor supply devoted to services corresponds to the real world phenomena of "deindustrialization".) Furthermore, the physical output of manufacturing grows perpetually while the services output remains constant. *But*, due to diminishing marginal utility and the ever-increasing relative scarcity of services, the relative price of services is ever-increasing (the celebrated Baumol-Bowen (1966) effect). These offsetting effects allow for *aggregate* labor's share to converge towards a constant, positive value.[4]

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[3] While this paper examines the dynamics of labor's share in response technical change, various theoretical and empirical research has explored the non-technical determinants of labor's share, e.g., Gomme and Greenwood (1995) and Boldrin and Horvath (1995) (unemployment insurance/labor contracts); Blanchard (1997), Blanchard and Wolfers (2000), Bentolia and Saint-Paul (2003) and Kessing (2003) (labor adjustment costs and bargaining power); Bertola (1993) (fiscal policy); and Ambler and Cardia (1998) (monopolistic competition).

[4] Hawtrey (1931, pp. 55-56) describes an uncannily similar story based on his observations of increased relative efficiency of some industries in the US: "There may be a general over-production of *factory* products. Modern methods of mass production tend to produce this result. Satiety of demand for such

This paper is organized as follows. Section II motivates our theory in terms of previous literature. Section III then develops our model. Sections IV and V discuss features of the steady-state and transition paths, respectively. Section VII concludes.

II. Technical Change, Induced Innovation, and Labor's Share

Given a growing capital to labor ratio, the relative stability of aggregate labor's share is remarkable because it is associated with the elasticity of output with respect to labor (Solow 1957). (Likewise, capital's share is associated with its elasticity.) The continuation of capital deepening while these elasticities remain constant strikes many as counterintuitive. Innovations to production possibilities are expected to be *labor-saving* (or *capital-using*) and, therefore, to raise capital's share of the aggregate product.[5]

Indeed, our model below is driven by endogenous labor-saving innovations, i.e., innovations increasing the optimal capital to labor ratio at constant factor prices. Consider a CES production function,

$$Y = A\left(\alpha K^{\phi} + (1-\alpha)L^{\phi}\right)^{\frac{1}{\phi}}, \tag{2.1}$$

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products might be reached, and the result might be the displacement of a large amount of redundant labor. But this does not happen *suddenly*. It has been happening visibly in the United States ever since [WWI]. The numbers employed in factories have been shrinking, not merely in proportion to population but absolutely [. . . .] Simultaneously the numbers employed in distribution and in rendering all the multifarious individual services that subserve [*sic*] the purposes of modern civilisation [*sic*] have been growing. It may be mentioned that in this division of tendencies agriculture is to be classed with manufacturing. Labor in agriculture is being displaced by machinery [. . . .] We may be approaching a state of society in which the mere production of any desired commodity becomes almost as easy and cheap as picking it up from the ground, and all the hard work will be put into the business of discovering the needs of consumers, specifying the appropriate products, and then (after they have been produced and transported) making them available for sale."

[5] Studies of labor-saving innovations in relation to growth and development constitute a substantial literature. Just a few examples are Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1966), Zeira (1998), Acemoglu (2002 and 2003) and Zuleta (2004). In addition, Boldrin and Levine (2002) present a model where new techniques are associated with new qualities of capital which are, in turn, labor-saving; Hornstein et al (2004) focus specifically on the role of capital-embodied technological change. Other theoretical reasons to expect the elasticity of output with respect to capital to be positively correlated with an economy's stage of development include the behavior of international trade (Hecksher (1919) and Ohlin (1939)) and capital flows (Dunning, 1998).

where the elasticity of substitution between capital (*K*) and labor (*L*) is $\sigma = (1-\phi)^{-1}$ and $-\infty < \phi \leq 1$. The optimal capital to labor ratio (*k*) is,

(2.2) $$k = \left[\frac{\alpha}{1-\alpha}\left(\frac{w}{r}\right)\right]^{\sigma}.$$

Thus, a labor-saving innovation is an increase in the distributional parameter, α, or the elasticity of substitution, σ.

For the special CES case of $\sigma = 1$ we have a Cobb-Douglas production function,

(2.1)' $$Y = AK^{\alpha}L^{1-\alpha}$$

and the optimal capital labor ratio is,

(2.2)' $$k = \frac{\alpha}{1-\alpha}\frac{w}{r}.$$

Labor saving innovation in a Cobb-Douglas economy is tantamount to an increase in α.[6]

Duffy and Papageorgiou (2000) and Durlauf and Johnson (1995) have provided evidence that these parameters, α and σ, have higher values in rich economies than in poor economies, so there is evidence that labor-saving innovation is part of economic development.

Many authors have argued that an increase in one factor relative to another will induce innovations favoring the abundant factor (e.g., Kennedy (1964), Kiley (1997), Krusell et al (2000) and Acemoglu (2003)). Early papers by Samuelson (1965) and Phelps and Drandakis (1966) attempted to reconcile this with the relative stability of aggregate labor's share. Both derived conditions resulting in purely labor-augmenting

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[6] All of the above can just as easily be applied to capital-saving innovations which decrease the optimal capital to labor ratio at fixed factor prices. However, as discussed below, because capital is reproducible and labor is a fixed factor (for the sake of discussing *per capita* production) incentives exist to focus primarily on labor-saving innovations.

technical change and, therefore, constant labor's share. Boldrin and Levine (2002) present a model similar in spirit, but where the factor-saving innovation involves the adoption of new, higher quality types of capital. Their model also provides an integration of long- and short-run phenomena, allowing for growth cycles where "recessions" are periods of technological adoption are periods of higher real wages and labor's share.[7]

The above models are fundamentally macroeconomic in the sense that all firms in the economy are essentially represented by aggregate production possibilities. In an important follow-up to these models, Acemoglu (2003) crafted an induced innovation model where numerous firms explicitly maximize profits, choosing to produce *either* capital- or labor-intensive intermediate goods. However, these firms only produce according to a single-factor (labor or capital) linear production function. Intermediate goods are then aggregated into a homogenous output via an aggregate production function. So the model cannot account for various industries with changing labor's shares and value-added shares that yield a stable aggregate labor's share. Our model is complementary to Acemoglu's and has the virtue that it can account for observations of labor's share, both aggregate and industry.

III. A Two-Sector Model of Growth

We build on the model of Zuleta (2005) where there are two sectors – one with a constant technology using only labor and one where both capital and labor are used and labor-saving innovations can be made. Each sector produces its own final good, and both goods are compliments in consumption. Capital and labor are broad categorizations

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[7] On the countercyclical nature of US aggregate labor's share see, e.g., Young (2004) and Boldrin and Horvath (1995). For further exploration of the potential for endogenous growth cycles as a result of labor-saving innovation see Boldrin and Fernández-Villaverde (2005).

meant to encompass, respectively, reproducible (e.g. both physical and human capital) and non-reproducible factors (e.g. raw labor and land).

We assume that there exists a set of technologies, differentiated by the elasticity of output with respect to capital, on the interval (0,1). At any instant every technology is available but the adoption of a technology (i.e., innovation) is costly. The cost to innovation is increasing in the capital intensity of the technology. The productivity of an innovation depends on the accumulated capital stock and, likewise, the productivity of the capital stock depends on the capital-intensity of the technology in place. This creates a tradeoff between investment in capital and capital-intensity.

Consider an economy with many identical agents and no population growth. There are no externalities in the model so we can speak of either a social planner or a representative agent (RA) solving the problem[8],

$$(3.1) \qquad max\int_0^\infty e^{-\rho t}\, log(C),$$

where $\rho > 0$, C is consumption, and we will omit the time arguments of variables for ease of expositional. Consumption is a Cobb-Douglas aggregate of two types of consumption goods,

$$(3.2) \qquad C = C_Y^{\lambda} C_X^{1-\lambda} \qquad 0 < \lambda < 1.$$

The RA is endowed with a single unit of labor at every instant.[9]

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[8] For a general model of endogenous growth under perfect competition see Boldrin and Levine (2005).

[9] The Cobb-Douglas assumption is for tractability. However, as stated below, we roughly think of Y and X as manufacturing and services, so this raises the validity of assuming a unitary elasticity of substitution between the two types of consumption. Stockman and Tesar (1995) estimated this elasticity to be 0.44 for G7 countries. However, including developing countries, Mendoza (1995) provided an estimate of 0.74 and, focusing entirely on developing countries, Ostry and Reinhart (1992) provided an estimate of 1.28. So there is considerable uncertainty surrounding this elasticity and some evidence that it is negatively related to the stage of development. (The above papers all phrase the question in terms of "tradable" and "non-tradable" goods.)

There are two productive sectors in the economy. The first uses labor to produce a good that is only useful for its consumption value. Production in this sector is,

$$X = C_X = BL_X \, , \tag{3.3}$$

where B is an efficiency parameter and L_X is the fraction of labor engaged in this sector's production. We (roughly) think of the output of this labor-intensive sector as services.[10]

The second sector uses both labor and capital (K) in production:

$$Y = C_Y + I = K^{\alpha}\left(AL_Y\right)^{1-\alpha} , \tag{3.4}$$

where I is investment and L_Y is the fraction of labor engaged in this sector's production. (Total instantaneous labor supply is assumed to be unity, $(1 - L_Y) = L_X$.) The good produced by the Y sector can be useful for both consumption and investment. We (roughly) think of this capital-intensive sector as manufacturing.

The investment from the Y sector can be devoted to capital deepening (i.e., increases in K) or it can be devoted to adopting more capital intensive production methods (i.e., increases in α). The intuition: to undertake labor-saving innovations some fraction of investment, $0 < (1 - \xi) < 1$, must be allocated towards the installation of new production methods, reorganization of existing productive structures, and replacement of obsolete capital. Considering K broadly, the increment $(1 - \xi)$ can also be thought of in terms of training for and adjustment to previously unused production methods.

The evolution of K as a function of the remaining fraction of investment, ξ, is standard:

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[10] The key difference between the two sectors is not that the first sector is labor-only, but rather the distinction made by Baumol (1967, pp. 415) of two types of production processes: those in which "labor is primarily an instrument" and those in which "labor is an end in itself." So, if we think of Y as manufactured goods and X as services, the important distinction is that labor is simply an input to manufacturing while it actually *is* a service.

(3.5) $$\dot{K} = \xi I \ .$$

Concerning α we assume that the entire spectrum of technologies, $\alpha = [0,1]$, is available at every instant. However, labor-saving innovations are costly in terms of I that cannot be devoted to increases in K. Specifically we assume that,

(3.6) $$\dot{\alpha} = (1-\alpha)(1-\xi)I \ .$$

The above specification embodies several desirable properties. First,

$$(1-\alpha)+\alpha = 1$$

so $(\dot{\alpha}+\alpha)$'s maximum value is unity: the maximum value consistent with constant returns to scale; second,

$$\frac{\partial \dot{\alpha}}{\partial \alpha} = -(1-\xi) < 0 \,,$$

so it becomes increasingly costly to increase α as it approaches unity; third,

$$\frac{\partial \dot{\alpha}}{\partial (1-\xi)} = (1-\alpha) \geq 0 \,;$$

so positive investment in installation/reorganization/replacement is never counterproductive; and

$$\dot{\alpha} = (1-1)(1-\xi) = 0 \quad \text{for} \qquad \alpha = 1,$$

so once the highest capital intensity has been reached there is no more innovation.[11]

At each instant the representative agent is confronted by the state of the economy (α and K) and makes choices (L_Y,, C_Y, I, and ξ).[12] The current-value Hamiltonian[13] is,

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[11] Seater (2005) presents a growth model that similarly has a Cobb-Douglas specification with an evolving parameter. However, the parameter evolution in Seater's model is exogenous; also his model is a one-sector model. Young (2004) considers parameter changes in the Cobb-Douglas specification of a real business cycle model.

[12] Because L_X is simply whatever labor remains after the allocation to L_Y, and because L_X determines C_X entirely, the representative agent's problem can be phrased entirely in terms of production and consumption of Y.

(3.7)

$$\begin{aligned}&H = \lambda \, log\,(C_Y) + (1-\lambda) log\,(B(1-L_Y))\\&\theta_1\left[\left(K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right)\xi\right] +\\&\theta_2\left[(1-\alpha)(1-\xi)\left(K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right)\right]\end{aligned}$$

where $\theta_1 = \pi_1 e^{\rho t}$ and $\theta_2 = \pi_2 e^{\rho t}$ and π_1 and π_2 are the shadow prices of capital and "capital-intensity." The first-order conditions for maximization are,

(3.8)

$$\frac{\partial \mathrm{H}}{\partial C_Y} = \frac{\lambda}{C_Y} - \theta_1\xi - \theta_2(1-\xi)(1-\alpha) = 0\,,$$

(3.9)

$$\begin{aligned}&\frac{\partial \mathrm{H}}{\partial L_Y} =\\&\theta_1\xi(1-\alpha)\left(\frac{K}{L_Y}\right)^{\alpha} A^{1-\alpha} + \theta_2(1-\alpha)^2(1-\xi)\left(\frac{K}{L_Y}\right)^{\alpha} A^{1-\alpha} - \left(\frac{1-\lambda}{1-L_Y}\right) = 0\end{aligned},$$

(3.10)

$$\frac{\partial \mathrm{H}}{\partial \xi} = \theta_1\left(K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right) - \theta_2(1-\alpha)\left(K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right) = 0\,,$$

(3.11)

$$\frac{\partial \mathrm{H}}{\partial K} + \dot{\theta}_1 = \theta_1\xi\alpha K^{\alpha-1}(AL_Y)^{1-\alpha} + \theta_2(1-\xi)(1-\alpha)\alpha K^{\alpha-1}(AL_Y)^{1-\alpha} + \dot{\theta}_1 = \rho\theta_1$$

(3.12)

$$\begin{aligned}&\frac{\partial \mathrm{H}}{\partial \alpha} + \dot{\theta}_2 =\\&[\theta_1\xi + \theta_2(1-\xi)(1-\alpha)]K^{\alpha}(AL_Y)^{1-\alpha} log\left(\frac{K}{AL_Y}\right) -,\\&\theta_2(1-\xi)\left(K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right) + \dot{\theta}_2 =\\&\rho\theta_2\end{aligned}$$

and we immediately note, from (3.10), that,

(3.13)

$$\theta_1 = \theta_2(1-\alpha_t)\,.$$

IV. Steady-State Properties

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[13] We ignore corner solutions for ease of exposition.

A defining property of this model is what value α converges to. Clearly the value will be some value within the 0 to 1 range. With the Cobb-Douglas production function,

$$(4.1) \qquad \alpha = (1-\alpha) K \, log\left(\frac{K}{AL_Y}\right) \quad \text{or} \quad \alpha = \frac{K \, log\left(\frac{K}{AL_Y}\right)}{1 + K \, log\left(\frac{K}{AL_Y}\right)} .$$

Totally differentiating (4.1) and manipulating leads to the expression,

$$(4.2) \qquad \dot{\alpha} = K(1-\alpha)^2 \left[\left(log\left(\frac{K}{AL_Y}\right) + 1 \right) \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right].$$

The equations (4.1) and (4.2) give us insights into the model's steady-state properties.

By (4.2), if $\dot{\alpha} = 0$ and $0 \leq \alpha < 1$, then it must be the case that,

$$(4.3) \qquad 0 = \left(log\left(\frac{K}{AL_Y}\right) + 1 \right) \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \quad \text{or} \quad K = 0.$$

A steady-state with $K = 0$ is possible and, if $\dot{L}_Y = 0$ in the long-run (which must be the case for $0 \leq L_Y \leq 1$), then it must also be the case that in general $\dot{K} = 0$.

On the other hand, if we allow that α converges to unity then (4.2) is still valid when $\dot{K} > 0$. Specifically, the optimal capital growth condition for this model is,[14]

$$(4.4) \qquad \frac{\dot{K}}{K} = \xi \left(K^{\alpha-1} (AL_Y)^{1-\alpha} - \frac{C_Y}{K} \right).$$

It can be demonstrated that the ratio of C_Y to K goes to ρ, so with $\alpha = 1$, $\dot{K} > 0$ as long as $\rho < 1$.

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[14] Detailed derivations of the optimal dynamic equations are provided in Appendix A.

So there are two basic types of long-run equilibrium that the model can support. First there is a *neoclassical steady-state* where $\frac{\dot{L}_Y}{L_Y}=\frac{\dot{\alpha}}{\alpha}=\frac{\dot{C}_X}{C_X}=\frac{\dot{C}_Y}{C_Y}=\frac{\dot{K}}{K}=0$, $0 \le \alpha < 1$, and $0 < L_Y < 1$; second there is *endogenous growth* where $\frac{\dot{L}_Y}{L_Y}=\frac{\dot{\alpha}}{\alpha}=\frac{\dot{C}_X}{C_X}=0$ and $\frac{\dot{C}_Y}{C_Y}=\frac{\dot{K}}{K}>0$, $\alpha = 1$, and $L_Y = 0$. In the former case, a true steady-state (in all the variables' levels) is achieved; in the later case, the production of Y in the economy collapses to "AK" and long-run economic growth is achieved.

Given these two long-run possibilities for capital intensity in the capital-using sector, what does this imply for aggregate labor's share (*LSH*)?[15]

$$LSH=\frac{(1-\alpha)}{(1-\alpha)+\alpha L_y} \tag{4.5}$$

In a neoclassical steady-state there are two determinants of *LSH*: α and L_Y. L_Y determines how labor is split between the Y sector (where labor's share is $1-\alpha$) and the X sector (where labor's share is always unity). The partial derivatives are,

$$\frac{\partial LSH}{\partial \alpha}=-\frac{L_y}{\left[(1-\alpha)+\alpha L_y\right]^2} \qquad \frac{\partial LSH}{\partial L_y}=-\frac{\alpha(1-\alpha)}{\left[(1-\alpha)+\alpha L_y\right]^2}, \tag{4.6}$$

which are both negative.

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[15] For calculating aggregate labor income and output for this economy, we must consider sectoral products valued in terms of their marginal utilities. So labor income is $\frac{\lambda}{C_y}(1-\alpha)K^{\alpha}A^{1-\alpha}L_y^{-\alpha}\bullet L_y+\frac{1-\lambda}{B(1-L_y)}B\bullet(1-L_y)$ and total income is $\frac{\lambda}{C_y}K^{\alpha}(AL_y)^{1-\alpha}+\frac{(1-\lambda)}{B(1-L_y)}B(1-L_y)$. Expression (4.5) is a simplification of the ratio of the two magnitudes.

When endogenous growth results, (4.5) cannot be evaluated at $\alpha = 1$ and $L_Y = 0$; rather, the limiting value must be considered. We can do this by exploiting the fact that Cobb-Douglas preferences imply,

$$\text{(4.7)} \qquad \frac{P_Y C_Y}{P_X C_X} = \frac{\lambda}{1-\lambda},$$

where P_Y and P_X are the prices of Y and X. As L_Y approaches 0, labor income approaches $P_X C_X$. LSH then becomes,

$$LSH = \frac{P_X C_X}{P_X C_X + P_Y\left(C_Y + I\right)} = \frac{1}{1 + \frac{P_Y C_Y}{P_X C_X} + \frac{P_Y I}{P_X C_X}} = \frac{1}{1 + \frac{\lambda}{1-\lambda} + \frac{\lambda}{1-\lambda}\frac{I}{C_Y}}.$$

which simplifies to,

$$\text{(4.9)} \qquad LSH = \frac{1-\lambda}{1 + \lambda \frac{I}{C_Y}}.$$

Given that I/C_Y is constant in the long-run, (4.9) is a constant between 0 and 1. Despite the growing capital to labor ratio during endogenous growth, LSH is stable and non-zero.

Also during endogenous growth, because $\dot{C}_Y > 0$ and $\dot{C}_X = 0$, by (4.7) P_X/P_Y must grow at the same rate as C_Y. So the economy displays an ever-increasing relative price of services – a widely-acknowledged feature of many real economies (e.g. De Gregorio et al (1994) and, most famously, Baumol and Bowen (1966)). This ever-increasing relative price of services, along with the ever-decreasing share of services in physical output, results in a constant long-run value-added share for services (SSH). Specifically, in the long-run

$$\text{(4.11)} \qquad SSH = \frac{1}{1 + \frac{\lambda}{1-\lambda}\frac{1}{\rho}} = LSH.$$

This connection between *SSH* and *LSH* is tied to share of total labor employed in the services sector going to unity.[16]

Finally we note that, during endogenous growth, the growth rates of C_Y and K are identical while $\frac{\dot{C}}{C} = \lambda \frac{\dot{C}_Y}{C_Y}$. Evaluating (4.4) at $\alpha = 1$ then implies that,

$$\text{(4.12)} \qquad \frac{\dot{C}_Y}{C_Y} = \frac{\dot{K}}{K} = 1 - \rho \qquad \text{and} \qquad \frac{\dot{C}}{C} = \lambda(1-\rho).$$

Aggregate consumption grows more slowly than the capital stock, broadly conceived.

V. Dynamics

In this section we elaborate on the transition of the model economy to either a neoclassical steady-state or long-run, endogenous growth path. Recall the expression for changes in α:

$$\text{(4.2)} \qquad \dot{\alpha} = K(1-\alpha)^2 \left[\left(log\left(\frac{K}{AL_Y} \right) + 1 \right) \frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y} \right].$$

The relationship between capital accumulation and the sectoral allocation of labor is fundamental to the dynamics of α. The expression for capital accumulation,

$$\text{(4.4)} \qquad \frac{\dot{K}}{K} = \xi \left(K^{\alpha - 1} (AL_Y)^{1-\alpha} - \frac{C_Y}{K} \right),$$

can be set against the expression for optimal sectoral labor growth,

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[16] Klyuev (2005) presents a model where constant but greater (relative to services) capital intensity in the manufacturing sector yields both the Baumol-Bowen effect and an increasing share of total labor employed in services. Based on the greater capital intensity, capital accumulation alone drives the results. Interestingly, Klyuev notes that models assuming that TFP grows faster in manufacturing than in services counterfactually predict a *decreasing* share of total labor employed in services. By incorporating labor-saving innovations, our two sector model reconciles the idea of faster technical change *and* a decreasing total labor employed share in manufacturing (i.e., "deindustrialization" – see Baumol et al (1989) and Rowthorn and Ramaswamy (1999)).

(5.1) $$\frac{\dot{L}_Y}{L_Y}=\frac{I}{K}\frac{(1-L_Y)}{(1-\alpha)}\left\{\frac{\left(\frac{\alpha}{K}+2-\alpha\right)\left(\alpha-K-\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right)+K}{\left\{\left(\frac{\alpha}{K}+2-\alpha\right)\left(L_Y+\frac{\alpha}{1-\alpha}\right)-(1-L_Y)K\right\}}\right\}.$$

Starting from any initial, positive ($\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho$), K grows and L_Y falls, both changes exerting negative influence on the marginal product of capital; by (4.2), α increases, exerting a positive influence on the marginal product of capital.

Like a standard growth model, diminishing returns are key and the incentives for investments (in both capital and capital intensity) vanish as ($\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho$) approaches zero. Whether the economy settles into a neoclassical steady-state or achieves endogenous growth depends on whether α converges to unity before ($\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho$) converges to 0.

We now employ the above dynamics to describe the evolution of *LSH*. Equation (4.5) can be rewritten as,

(5.2) $$LSH=\frac{1}{1+\frac{\alpha}{1-\alpha}L_Y}.$$

This expression implies that the dynamics of *LSH* depend on the sign of $\frac{\dot{\alpha}}{\alpha}+\frac{\dot{\alpha}}{1-\alpha}+\frac{\dot{L}_Y}{L_Y}$.

It can be demonstrated that this expression is always non-negative.[17] Starting from below the economy's steady-state/balanced growth path, aggregate labor's share converges to its long-run, constant value from above. This is not inconsistent with the pattern of US

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[17] The proof of this claim is in Appendix C. Some claims below that are also left unproved in the text are demonstrated in previous and subsequent Appendices.

labor's share pictured in **Figure 1**.[18] Furthermore, during the transition labor's share in the *Y* sector falls while it remains constant (at unity) in the *X* sector, so that the sectoral labor's shares are uncorrelated over time. This is the case despite the fundamental role that the *X* sector, with its time-invariant labor's share, plays in preventing aggregate labor's share from going to zero.

Despite *LSH*'s transitional decrease, the *X* sector's share of the economy's output increases. The share of services is,

$$XSH = \frac{P_X C_X}{P_X C_X + P_Y (C_Y + I)}, \tag{5.5}$$

which is notably identical to the expression for *LSH* during long-run, endogenous growth (but *not* for *LSH* in general). Expression (5.5) can, as in section IV, be manipulated into,

$$XSH = \frac{1-\lambda}{1+\lambda \frac{I}{C_Y}} . \tag{5.6}$$

It can be shown that I/C_Y decreases during the transition, so *XSH* increases.

VII. Conclusions

The process of economic development is famously characterized by certain great macroeconomic ratios, e.g. capital to output ratios, the return to capital, and – the motivation for this paper – labor's share. These ratios display surprising stability that apparently transcends stages of development. In the case of aggregate labor's share, the

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[18] Taking a longer, historical view, this is not in conflict with the lack of observable trend in US aggregate labor's share. Either (a) the US attained the neighborhood of balanced growth long ago, so the point is moot, or (b) if the transition has indeed been very long – say 200 years – then, beginning from a labor's share near unity, the decrease to a constant value around 65 percent would be extremely slow, obscuring the downward trend. Beyond the US, evidence is mixed on this point. Torrini (2005) reports that Italy's labor's share declined from the mid-1970s through the mid-1990s; Garrido Ruiz (2005) reports that Spain's labor's share increased from 1955 through 2005; Gollin (2002) suggests that there is no relationship between the level of labor's share and the level of economic development.

relative stability is all the more surprising given the flux of individual industries' labor's shares and their share's in aggregate output. The challenge is to account for labor's shares, both aggregate and industry.

In this paper we develop a two-sector model of economic development in the spirit of the induced innovation literature. One sector allows for innovations that increase the capital intensity of production, naturally raising capital's share in that sector's physical product. However, the second (labor-intensive) sector maintains a constant marginal physical product of labor, attracting an increasing portion of the available labor supply. Because the labor-intensive sector can have no long-run growth in physical product, the relative price of its output increases over time, so the marginal value product of labor increases. This effect maintains a non-zero labor's share even when the innovative sector achieves long-run, "AK"-type growth in physical product.

Our model provides a framework for interpreting several empirical regularities of real economies: (i) manufacturing industries becoming increasingly capital-intensive over time despite (ii) an increase in the relative price and share in total output of service industries; (iii) aggregate labor's share displaying a horizontal trend despite (iv) individual industry labor's shares that are uncorrelated with one another. Furthermore, because the model can attain a neoclassical steady-state or long-run, endogenous growth, it has the potential to account for a wide range of real world development experiences.

It would be interesting to extend the model so that both sectors can use capital in production, but one retains significantly more innovative potential. (This would certainly be a more realistic contrast between manufactured goods and services production.) It would also be interesting to simulate a calibrated version of the model to quantify the magnitude and duration of aggregate labor's share transition to its long-run value. As well, given the coincidence of growth in services' share of the US economy and greater macroeconomic stability, the present model could be modified into a real business cycle model to explore whether the correlation actually signals causation.

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Tables

Table 1 – Decomposition of US Aggregate Labor's Share Changes

	Labor's Share Change Component		
Statistic	Within-Industry	Between-Industry	Covariance
Mean	-0.000	-0.000	-0.000
σ	0.010	0.002	0.000
σ/σ_{Actual}	1.079	0.252	0.046
$\rho_{Component,Actual}$	0.967	-0.148	-0.072

Notes: Decomposition from Young (2005) and based on the method by Foster et al (2001). σ denotes standard deviation. ρ denotes correlation. "*Actual*" refers to actual changes in aggregate labor's share.

Table 2 – Summary Statistics for Three US Industry Groups

Statistic for	Agriculture	Manufacturing	Services
Labor's Share			
Mean	0.645	0.722	0.689
σ	0.066	0.021	0.035
$\rho_{x,Agriculture}$	1.000	0.235	-0.601
$\rho_{x,Manufacturing}$	0.235	1.000	-0.215
$\rho_{x,Services}$	-0.601	-0.215	1.000
$\Delta_{1958,1996}$	-0.203	-0.079	0.074
Value-Added Share			
Mean	0.034	0.285	0.175
σ	0.009	0.023	0.040
$\rho_{x,Agriculture}$	1.000	0.758	-0.870
$\rho_{x,Manufacturing}$	0.758	1.000	-0.943
$\rho_{x,Services}$	-0.870	-0.943	1.000
$\Delta_{1958,1996}$	-0.034	-0.045	0.128

Notes: Data from 35-KLEM database. Methodology described in Jorgenson et al (1987). Manufacturing includes "Food and Kindred Products," Tobacco," "Textile Mill Products," "Apparel," "Limber and Wood," "Furniture and Fixtures," "Paper and Allied," "Chemicals," "Petroleum and Coal Products," "Rubber and Miscellaneous Products," "Leather," "Stone, Clay and Glass," "Primary Metal," "Fabricated Metal," "Non-electrical," "Motor Vehicle," "Transportation Equipment and Ordinance," "Instruments," and "Miscellaneous Manufacturing" industries.

TABLE 3–CHANGE IN INDUSTRY LABOR'S AND VALUE-ADDED SHARES AND INDUSTRY CONTRIBUTIONS TO AGGREGATE LABOR'S SHARE CHANGE: 1958 – 1996

Industry	Description	Change in Labor's Share	Change in Value-Added Share	Contribution to Agg. Labor's Share Change
1	Agriculture	-0.203	-0.034	-0.031
2	Metal Mining	-0.027	-0.001	-0.001
3	Coal Mining	-0.095	-0.002	-0.002
4	Oil and Gas Extraction	-0.032	-0.007	-0.003
5	Non-metallic Mining	-0.124	-0.001	-0.001
6	Construction	0.022	-0.020	-0.016
7	Food & Kindred Products	-0.187	-0.006	-0.008
8	Tobacco	-0.160	0.001	0.000
9	Textile Mill Products	-0.066	-0.003	-0.003
10	Apparel	-0.093	-0.008	-0.008
11	Limber and Wood	-0.069	-0.002	-0.002
12	Furniture and Fixtures	-0.064	-0.001	-0.001
13	Paper and Allied	-0.046	-0.002	-0.002
14	Print., Publishing & Allied	-0.042	0.002	0.001
15	Chemicals	-0.034	0.005	0.002
16	Petroleum & Coal Products	-0.206	0.002	0.000
17	Rubber & Misc. Prod.	0.001	0.003	0.002
18	Leather	-0.364	-0.003	-0.003
19	Stone, Clay, Glass	0.097	-0.006	-0.003
20	Primary Metal	0.104	-0.016	-0.010
21	Fabricated Metal	-0.177	-0.008	-0.009
22	Non-electrical Industry	-0.040	0.001	-0.001
23	Electrical Industry	-0.179	0.004	-0.001
24	Motor Vehicles	0.018	-0.001	-0.001
25	Transp. Equip & Ord.	0.023	-0.008	-0.007
26	Instruments	0.023	0.004	0.004
27	Misc. Manufacturing	-0.216	-0.002	-0.002
28	Transportation	0.064	-0.019	-0.010
29	Communications	-0.062	0.003	0.000
30	Electrical Utilities	0.010	-0.002	-0.001
31	Gas Utilities	-0.061	-0.002	-0.001
32	Trade	0.000	-0.044	-0.034
33	Fin., Ins. & Real Estate	0.006	0.033	0.015
34	Services	0.074	0.128	0.104
35	Government Enterprises	-0.158	0.014	0.005

Notes: Calculated from 35 annual industries' data, 1958 – 1996. Labor's share is that of annual value added. Aggregate labor's share is calculated as a weighted average of industry labor's shares with industry shares in total value-added as weights.

FIGURES

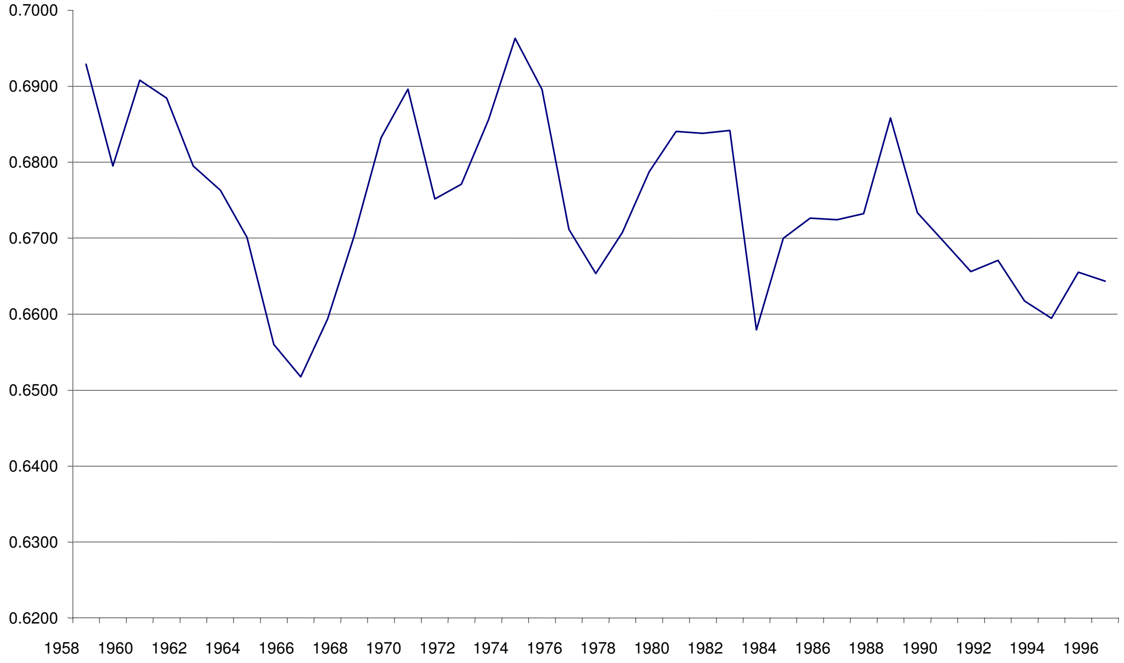

FIGURE 1. US AGGREGATE LABOR'S SHARE: 1958 - 1996

Notes: Calculated from aggregation of 35 industries' data. At the industry level, calculations are of labor's share of value added. At the aggregate level, industries weighted by their share of total value added.

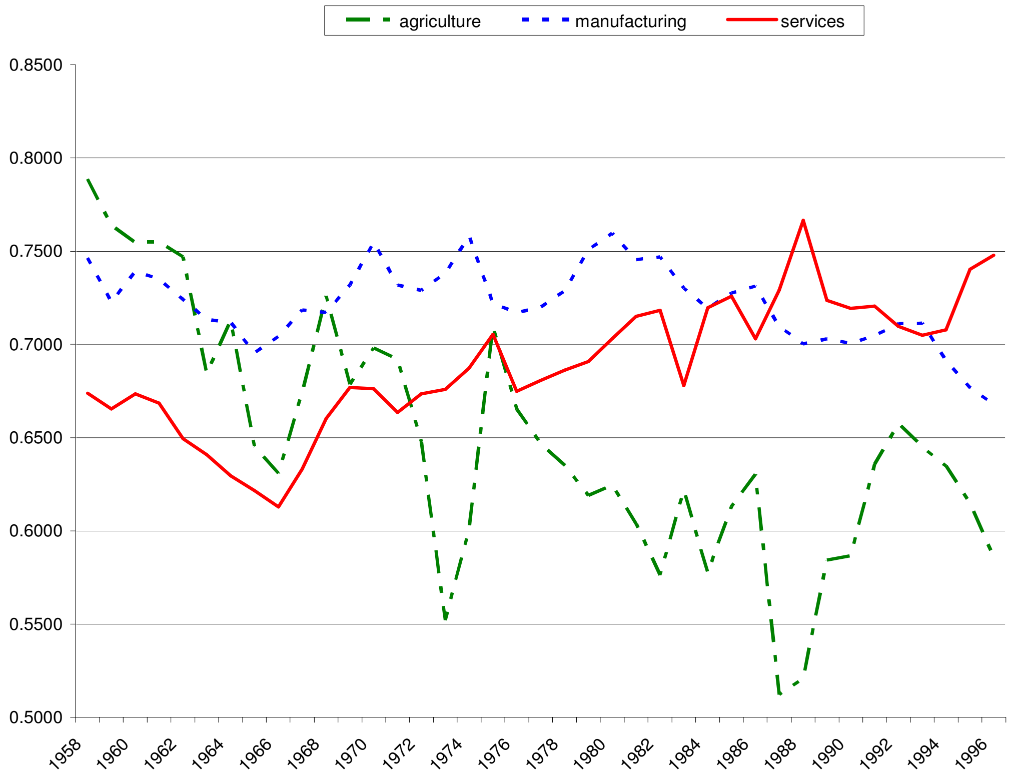

FIGURE 2A. SELECT MAJOR US INDUSTRY LABOR'S SHARES

Notes: Data from 35-KLEM database. Methodology described in Jorgenson et al (1987). Manufacturing includes "Food and Kindred Products," Tobacco," "Textile Mill Products," "Apparel," "Limber and Wood," "Furniture and Fixtures," "Paper and Allied," "Chemicals," "Petroleum and Coal Products," "Rubber and Miscellaneous Products," "Leather," "Stone, Clay and Glass," "Primary Metal," "Fabricated Metal," "Non-electrical," "Motor Vehicle," "Transportation Equipment and Ordinance," "Instruments," and "Miscellaneous Manufacturing" industries.

FIGURES (CONT.)

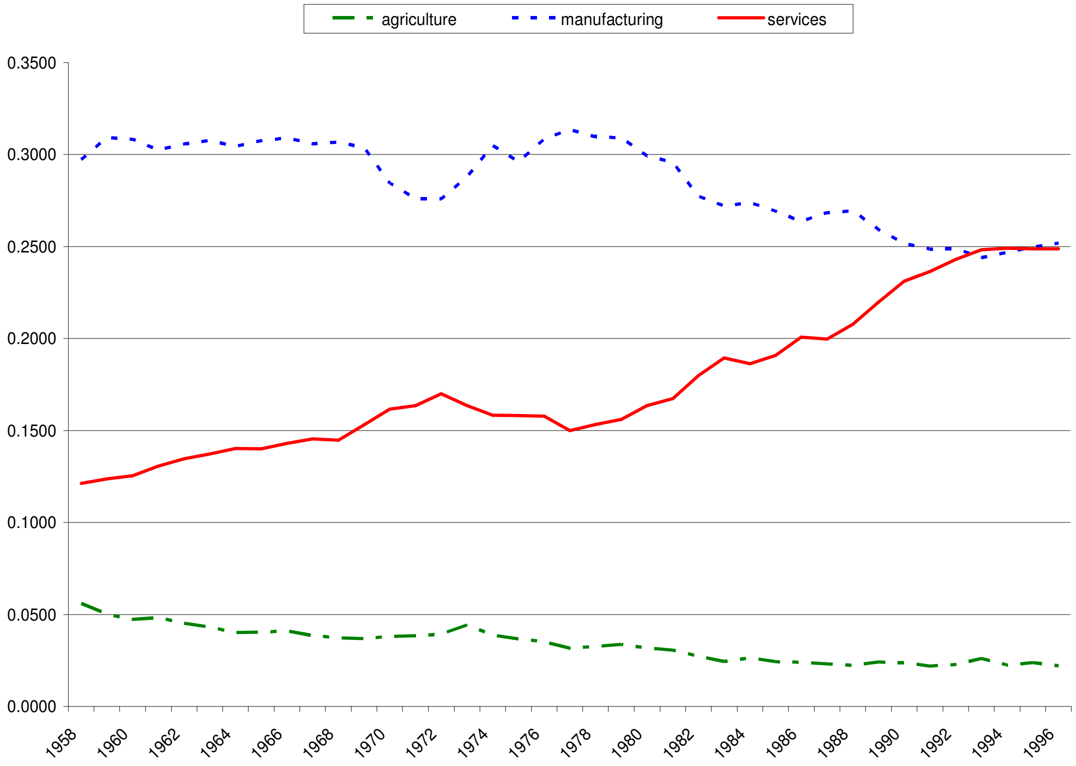

FIGURE 2B. SELECT MAJOR US INDUSTRY VALUE-ADDED SHARES

Notes: Data from 35-KLEM database. Methodology described in Jorgenson et al (1987). Manufacturing includes "Food and Kindred Products," Tobacco," "Textile Mill Products," "Apparel," "Limber and Wood," "Furniture and Fixtures," "Paper and Allied," "Chemicals," "Petroleum and Coal Products," "Rubber and Miscellaneous Products," "Leather," "Stone, Clay and Glass," "Primary Metal," "Fabricated Metal," "Non-electrical," "Motor Vehicle," "Transportation Equipment and Ordinance," "Instruments," and "Miscellaneous Manufacturing" industries.

Appendix A: Model Derivations

The basic framework of the model consists of,

$$max \int_{0}^{\infty} e^{-\rho t} \, log(C) \qquad \rho > 0, \tag{A.1}$$

$$C = C_Y^{\lambda} C_X^{1-\lambda} \qquad 0 < \lambda < 1, \tag{A.2}$$

$$X = C_X = BL_X \,, \tag{A.3}$$

$$Y = C_Y + I = K^{\alpha}\left(AL_Y\right)^{1-\alpha} \qquad (1 - L_Y) = L_X, \tag{A.4}$$

$$\dot{K} = \xi I \qquad 0 < \xi < 1, \tag{A.5}$$

and

$$\dot{\alpha} = (1-\alpha)(1-\xi) \,. \tag{A.6}$$

The current-value Hamiltonian is,

$$\begin{aligned} H &= \lambda \, log\,(C_Y) + (1-\lambda) log\,(B(1-L_Y)) \\ &\theta_1 \left[\left(K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right)\xi\right] + \\ &\theta_2 \left[(1-\alpha)(1-\xi)\left(K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right)\right] \end{aligned} \,. \tag{A.7}$$

The necessary condition derived from the Hamiltonian are,

$$\frac{\partial H}{\partial C_Y} = \frac{\lambda}{C_Y} - \theta_1 \xi - \theta_2 (1-\alpha)(1-\xi) = 0 \,, \tag{A.8}$$

$$\frac{\partial H}{\partial L_Y} = \theta_1 \xi (1-\alpha)\left(\frac{K}{L_Y}\right) A^{1-\alpha} + \theta_2 (1-\xi)(1-\alpha)^2 \left(\frac{K}{L_Y}\right)^{\alpha} A^{1-\alpha} - \left(\frac{1-\lambda}{1-L_Y}\right) = 0 \,, \tag{A.9}$$

$$\frac{\partial H}{\partial \xi} = \theta_1 \left[K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right] - \theta_2 (1-\alpha)\left[K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right] = 0 \,, \tag{A.10}$$

$$\begin{aligned} &\frac{\partial H}{\partial K} + \dot{\theta}_1 = \\ &\theta_1 \xi \alpha K^{\alpha-1}(AL_Y)^{1-\alpha} + \theta_2 (1-\alpha)(1-\xi)\alpha K^{\alpha-1}(AL_Y)^{1-\alpha} + \dot{\theta}_1 = \rho \theta_1 \end{aligned} \,, \tag{A.11}$$

and

$$\frac{\partial H}{\partial \alpha} + \dot{\theta}_2 =$$

(A.12) $$\begin{aligned} &\theta_1 \xi K^{\alpha} (AL_Y)^{1-\alpha}\, log\left(\frac{K}{AL_Y}\right) + \theta_2 (1-\alpha)(1-\xi) K^{\alpha} (AL_Y)^{1-\alpha}\, log\left(\frac{K}{AL_Y}\right) - . \\ &\theta_2 (1-\xi)\left[K^{\alpha}(AL_Y)^{1-\alpha} - C_Y\right] + \dot{\theta}_2 = \rho \theta_2 \end{aligned}$$

From (A.10) we have $\theta_1 = \theta_2(1 - \alpha)$ so that (A.8) can be rewritten,

$$\frac{\lambda}{C_Y} = \theta_1 = \theta_2 (1-\alpha).$$

Differentiating with respect to time:

$$\frac{\dot{C}_Y}{C_Y} = -\frac{\dot{\theta}_1}{\theta_1}.$$

From (A.9):

$$\left[\theta_1 \xi (1-\alpha) + \theta_2 (1-\xi)(1-\alpha)^2\right]\left(\frac{K}{L_Y}\right)^{\alpha} A^{1-\alpha} = \left(\frac{1-\lambda}{1-L_Y}\right).$$

Combining the above with $\theta_1 = \theta_2(1 - \alpha)$:

(A.13) $$\begin{aligned} \theta_1 &= \left(\frac{1-\lambda}{1-L_Y}\right)\left[(1-\alpha)\left(\frac{K}{L_Y}\right)^{\alpha} A^{1-\alpha}\right]^{-1} \\ \theta_2 &= \left(\frac{1-\lambda}{1-L_Y}\right)\left[(1-\alpha)^2\left(\frac{K}{L_Y}\right)^{\alpha} A^{1-\alpha}\right]^{-1}. \end{aligned}$$

Differentiating with respect to time:

$$\begin{aligned} \frac{\dot{\theta}_1}{\theta_1} &= \frac{\dot{L}_Y}{1-L_Y} + \alpha\left(\frac{\dot{L}_Y}{L_Y} - \frac{\dot{K}}{K}\right) + \frac{\dot{\alpha}}{1-\alpha} - \dot{\alpha}\, log\left(\frac{K}{AL_Y}\right) \\ \frac{\dot{\theta}_2}{\theta_2} &= \frac{\dot{L}_Y}{1-L_Y} + \alpha\left(\frac{\dot{L}_Y}{L_Y} - \frac{\dot{K}}{K}\right) + 2\left(\frac{\dot{\alpha}}{1-\alpha}\right) - \dot{\alpha}\, log\left(\frac{K}{AL_Y}\right) \end{aligned}$$

Combining (A.13) with $\frac{\lambda}{C_Y} = \theta_1 = \theta_2(1-\alpha)$:

(A.14) $$(1-L_Y)=\left(\frac{1-\lambda}{\lambda}\right)\frac{C_Y}{(1-\alpha)\left(\frac{K}{L_Y}\right)^{\alpha}A^{1-\alpha}}\ .$$

Differentiating with respect to time:

$$-\dot{L}_Y=\left[(1-\alpha)\left(\frac{K}{L_Y}\right)^{\alpha}A^{1-\alpha}\right]^{-1}\left(\frac{1-\lambda}{\lambda}\right)\dot{C}_Y-\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\lambda}{\lambda}\right)C_Y K^{-\alpha-1}L_Y^{\alpha}A^{\alpha-1}\dot{K}+$$
$$\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\lambda}{\lambda}\right)C_Y K^{-\alpha}(L_Y A)^{\alpha-1}\dot{L}_Y+\left(\frac{1-\lambda}{\lambda}\right)C_Y\left[\frac{\partial\left\{(1-\alpha)\left(\frac{K}{L_Y}\right)^{\alpha}A^{1-\alpha}\right\}}{\partial\alpha}\right]\dot{\alpha}\ ,$$

where,

$$\frac{\partial\left\{(1-\alpha)\left(\frac{K}{L_Y}\right)^{\alpha}A^{1-\alpha}\right\}}{\partial\alpha}=-\left[\frac{(1-\alpha)log\left(\frac{K}{AL_Y}\right)-1}{(1-\alpha)^2\left(\frac{K}{AL_Y}\right)^{\alpha}A^{1-\alpha}}\right].$$

Putting the above into growth rates and exploiting the fact that, by (A.14),

$$\left(\frac{\lambda}{1-\lambda}\right)\left(\frac{(1-\alpha)\left(\frac{K}{L_Y}\right)^{\alpha}A^{1-\alpha}}{C_Y}\right)=\frac{1}{1-L_Y},$$

we arrive at,

(A.15) $$\frac{\dot{C}_Y}{C_Y}=\alpha\left[\frac{\dot{K}}{K}-\frac{\dot{L}_Y}{L_Y}\right]+\left[log\left(\frac{K}{AL_Y}\right)-\left(\frac{1}{1-\alpha}\right)\right]\dot{\alpha}+\frac{\dot{L}_Y}{(1-L_Y)}\,.$$

Combing $\theta_1=\theta_2(1-\alpha)$ with (A.11) yields, (A.16) $$-\frac{\dot{\theta}_1}{\theta_1} = \alpha K^{\alpha-1}\left(AL_Y\right)^{1-\alpha} - \rho\,,$$

or,

(A.17) $$\frac{\dot{C}_Y}{C_Y} = \alpha K^{\alpha-1}\left(AL_Y\right)^{1-\alpha} - \rho\,.$$

From (A.15) and the time-differentiation of (13) comes,

$$\frac{\dot{L}_Y}{1-L_Y} = \alpha\left(\frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y}\right) - \frac{\dot{\alpha}}{1-\alpha} + \dot{\alpha}\,log\left(\frac{K}{AL_Y}\right) - \left[\alpha K^{\alpha-1}\left(AL_Y\right)^{1-\alpha} - \rho\right],$$

which, using (A.5) and (A.6), then becomes,

(A.18) $$\frac{\dot{L}_Y}{1-L_Y} = \alpha\left(\frac{\xi I}{K} - \frac{\dot{L}_Y}{L_Y}\right) - (1-\xi)I + (1-\xi)(1-\alpha)I\,log\left(\frac{K}{AL_Y}\right) - \left[\alpha K^{\alpha-1}\left(AL_Y\right)^{1-\alpha} - \rho\right]$$

Combing $\theta_1 = \theta_2(1-\alpha)$ with (A.12) yields,

(A.19) $$-\frac{\dot{\theta}_2}{\theta_2} = -\frac{\dot{\alpha}}{1-\alpha} + (1-\alpha)K^{\alpha}\left(AL_Y\right)^{1-\alpha}\,log\left(\frac{K}{AL_Y}\right) - \rho\,,$$

implying, with (A.16), that,

$$(1-\alpha)K^{\alpha}\left(AL_Y\right)^{1-\alpha}\,log\left(\frac{K}{AL_Y}\right) = \alpha K^{\alpha-1}\left(AL_Y\right)^{1-\alpha},$$

or,

(A.20) $$\alpha = \frac{K\,log\left(\frac{K}{AL_Y}\right)}{1 + K\,log\left(\frac{K}{AL_Y}\right)}\,.$$

Rearranging (A.18) and using (A.20) yields, (A.21) $$\frac{\dot{L}_Y}{L_Y}=\left[\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right]\left[\alpha\left(\frac{I}{K}\right)-(1-\xi)I-\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right].$$

Differentiating (A.20) with respect to time:

(A.22) $$\dot{\alpha}=K(1-\alpha)^2\left\{\left[log\left(\frac{K}{AL_Y}\right)+1\right]\left(\frac{\dot{K}}{K}\right)-\frac{\dot{L}_Y}{L_Y}\right\}.$$

Combining (A.22) with (A.6) and rearranging:

(A.23) $$\frac{\dot{L}_Y}{L_Y}=\left(\frac{I}{K}\right)\left\{\left[1+log\left(\frac{K}{AL_Y}\right)\right]\xi-\left(\frac{1-\xi}{1-\alpha}\right)\right\}.$$

Using (A.21) and (A.23) along with (A.5) and (A.6) results in, with good deal of rearranging,

(A.24) $$\xi=\frac{(1-\alpha)(1-L_Y)\left[\alpha-K-\left(\frac{K}{I}\right)\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right]+L_Y(1-\alpha)+\alpha}{\left[L_Y(1-\alpha)+\alpha\right]\left[\left(\frac{\alpha}{K}+2-\alpha\right)-\left(\frac{(1-\alpha)(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)K\right]}.$$

Appendix B: Claim: $\xi \geq \alpha$.

From (A.24) we can state that,

$$\text{(B.1)} \qquad \xi = \frac{\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(\alpha - K - \frac{1}{I}\left(\alpha K^{\alpha}(AL_Y)^{1-\alpha} - \rho K\right)\right) + \frac{1}{1-\alpha}}{\left\{\left(\frac{\alpha}{1-\alpha}\frac{1}{K}+1\right) + \frac{1}{(1-\alpha)} - \left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)K\right\}}.$$

So,

$$\text{(B.2)} \qquad \frac{\alpha}{\xi} = \frac{\left(\frac{\alpha}{1-\alpha}\frac{\alpha}{K}+\alpha\right) + \frac{\alpha}{(1-\alpha_t)} - \left(\frac{(1-L)\alpha}{L_Y(1-\alpha)+\alpha}\right)K}{\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(\alpha - K - \frac{1}{I}\left(\alpha K^{\alpha}(AL_Y)^{1-\alpha} - \rho K\right)\right) + \frac{1}{1-\alpha}}.$$

Call A: numerator and B: denominator.

Claim B1: ***If B>0 and A>0 then B>A.***

If B > 0 then

$$\text{(B.3)} \qquad \left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right) + \frac{1}{1-\alpha} > \left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K + \frac{1}{I}\left(\alpha K^{\alpha}(AL_Y)^{1-\alpha} + \rho K\right)\right).$$

If A > 0 then

$$\text{(B.4)} \qquad \left(\frac{\alpha}{1-\alpha}\frac{\alpha}{K}+\alpha\right) + \frac{\alpha}{(1-\alpha)} > \left(\frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha}\right)K .$$

From (B.3):

$$\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right) + \frac{1}{1-\alpha} > \left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K + \frac{1}{I}\left(\alpha K^{\alpha}(AL_Y)^{1-\alpha} - \rho K\right)\right).$$

Subtracting $\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right) + \frac{\alpha}{(1-\alpha)}$ from both sides yields.

(B.5)

$$\begin{aligned}&\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)+\frac{1}{1-\alpha}-\alpha-\frac{\alpha}{1-\alpha}-\frac{\alpha^2}{1-\alpha}\frac{1}{K}>\\&-\left[\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}\right]+\\&\left[\left(\frac{\alpha}{1-\alpha}K+\frac{1}{\alpha}\right)+\frac{1}{(1-\alpha)}\right]\left(1+\frac{1}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right)\end{aligned}.$$

From (B.4), $\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}>\left(\frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha}\right)K$, so,

$$\begin{aligned}&-\left[\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}\right]+\\&\left(\frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)\left(K+\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right)\\&<\left(\frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)\left(K(1-\alpha)+\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right)\end{aligned}.$$

Thus,

$$\begin{aligned}&\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)+\frac{1}{1-\alpha}-\alpha-\frac{\alpha}{1-\alpha}-\frac{\alpha^2}{1-\alpha}\frac{1}{K}>\\&\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)K\left((1-\alpha)+\frac{1}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}+\rho\right)\right)\end{aligned},$$

and,

$$\begin{aligned}&\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)K-\frac{\alpha}{1-\alpha}-\alpha-\frac{\alpha^2}{1-\alpha}\frac{1}{K}>\\&\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K-\alpha+\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}+\rho\right)\right)-\frac{1}{1-\alpha}\end{aligned}.$$

Finally,

(B.6)
$$\frac{1}{1-\alpha}\frac{\alpha}{1-\alpha}+\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha-\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)K<$$
$$\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(\alpha-K-\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}+\rho\right)\right)+\frac{1}{1-\alpha}.$$

Therefore,

(B.7)
$$\frac{\alpha}{\xi}=\frac{\left(\frac{\alpha}{1-\alpha}\frac{\alpha}{K}+1\right)+\frac{\alpha}{(1-\alpha)}-\left(\frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha}\right)K}{\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(\alpha-K-\frac{1}{I}\left(\alpha K^{\alpha}(AL_{Yt})^{1-\alpha}-\rho K\right)\right)+\frac{1}{1-\alpha}}<1.$$

Claim B2: ***If B<0 and A<0 then B<A.***

Proof:

If B < 0 then,

(B.8)
$$\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)+\frac{1}{1-\alpha}<\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K+\frac{1}{I}\left(\alpha K^{\alpha}(AL_Y)^{1-\alpha}+\rho K\right)\right).$$

If A < 0 then,

(B.9)
$$\left(\frac{\alpha}{1-\alpha}\frac{\alpha}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}<\left(\frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha}\right)K.$$

From (B.8):

$$\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)+\frac{1}{1-\alpha}<\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K+\frac{1}{I}\left(\alpha K^{\alpha}(AL_Y)^{1-\alpha}-\rho K\right)\right).$$

Subtracting $\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}$ from both sides yields,

(B.10)

$$\begin{aligned}&\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)+\frac{1}{1-\alpha}-\alpha-\frac{\alpha}{1-\alpha}-\frac{\alpha^2}{1-\alpha}\frac{1}{K}<\\&-\left[\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}\right]+\\&\left[\left(\frac{\alpha}{1-\alpha}K+\frac{1}{\alpha}\right)+\frac{1}{(1-\alpha)}\right]\left(1+\frac{1}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right)\end{aligned}.$$

From (B.9), $\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha_t)}<\left(\frac{(1-L)\alpha}{L(1-\alpha)+\alpha}\right)K$, so,

$$\begin{aligned}&-\left[\left(\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}\right]+\\&\left(\frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)\left(K+\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right)\\&>\left(\frac{(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)\left(K(1-\alpha)+\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}-\rho\right)\right)\end{aligned}.$$

Thus,

$$\begin{aligned}&\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)+\frac{1}{1-\alpha}-\alpha-\frac{\alpha}{1-\alpha}-\frac{\alpha^2}{1-\alpha}\frac{1}{K}<\\&\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)K\left((1-\alpha)+\frac{1}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}+\rho\right)\right)\end{aligned},$$

and,

$$\begin{aligned}&\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)K-\alpha-\frac{\alpha}{1-\alpha}-\frac{\alpha^2}{1-\alpha}\frac{1}{K}<\\&\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K-\alpha+\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}+\rho\right)\right)-\frac{1}{1-\alpha}\end{aligned}.$$

Finally,

$$\frac{\alpha^2}{1-\alpha}\frac{1}{K}+\alpha+\frac{\alpha}{1-\alpha}-\left(\frac{\alpha(1-L_Y)}{L_Y(1-\alpha)+\alpha}\right)K > \frac{1}{1-\alpha}-\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(K-\alpha+\frac{K}{I}\left(\alpha K^{\alpha-1}(AL_Y)^{1-\alpha}+\rho\right)\right). \tag{B.11}$$

Therefore,

$$\frac{\alpha}{\xi}=\frac{\left(\frac{\alpha}{1-\alpha}\frac{\alpha}{K}+\alpha\right)+\frac{\alpha}{(1-\alpha)}-\left(\frac{(1-L_Y)\alpha}{L_Y(1-\alpha)+\alpha}\right)K}{\left(\frac{1-L_Y}{L_Y(1-\alpha)+\alpha}\right)\left(\alpha-K-\frac{1}{I}\left(\alpha K^{\alpha}(AL_Y)^{1-\alpha}-\rho K\right)\right)+\frac{1}{1-\alpha}}<1. \tag{B.12}$$

Having demonstrated **B1** and **B2**, the general claim that $\xi \geq \alpha$ is established.

Appendix C: Claim: *Labor's share converges from above to a positive value.*

Labor's share is,

(C.1) $$LSH = \frac{w_Y L_Y + w_X L_X}{P_Y Y + P_X X} \; .$$

Since the real wages (in terms of marginal utility units) are equated across sectors and total labor supply is unity we have,

(C.2) $$LSH = \frac{\left(\frac{1-\lambda}{C_X}\right)B}{\frac{\lambda}{C_Y} K^{\alpha}\left(AL_Y\right)^{1-\alpha} + \frac{1-\lambda}{C_X} BL_X},$$

total and consumption output from the *X* sector are identical:

(C.3) $$LSH = \frac{\left(\frac{1-\lambda}{1-L_Y}\right)}{\frac{\lambda}{C_Y} K^{\alpha}\left(AL_Y\right)^{1-\alpha} + (1-\lambda)} = \frac{\left(\frac{1}{1-L_Y}\right)}{1+\frac{\lambda}{1-\lambda}\frac{K^{\alpha}\left(AL_Y\right)^{1-\alpha}}{C_Y}} \; .$$

Since $\frac{1}{1-L_Y}$ unambiguously decreases during the transition, if we can demonstrate that $\frac{K^{\alpha}\left(AL_Y\right)^{1-\alpha}}{C_Y}$ increases during the transition, *LSH* will unambiguously decrease towards its steady-state value of $LSH = \frac{1}{1+\frac{\lambda}{1-\lambda}\frac{1}{\rho}}$. The growth rate of the relevant ratio is,

(C.4) $$\alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}_Y}{L_Y} + log\left(\frac{K}{AL_Y}\right)\dot{\alpha} - \frac{\dot{C}_Y}{C_Y} \; .$$

Furthermore, if we combine (C.4) with (A.15), (A.15) $$\frac{\dot{C}_Y}{C_Y} = \alpha\left[\frac{\dot{K}}{K} - \frac{\dot{L}_Y}{L_Y}\right] + \left[log\left(\frac{K}{AL_Y}\right) - \left(\frac{1}{1-\alpha}\right)\right]\dot{\alpha} + \frac{\dot{L}_Y}{(1-L_Y)},$$

we arrive at,

(C.5) $$\begin{gathered}\frac{\dot{L}_Y}{L_Y} + \frac{\dot{\alpha}}{1-\alpha} - \frac{\dot{L}_Y}{1-L_Y} = \\ \frac{\dot{\alpha}}{1-\alpha} + \left\{\frac{1}{L_Y} - \frac{1}{1-L_Y}\right\}\dot{L}_Y\end{gathered}.$$

We know that $\frac{\dot{\alpha}}{1-\alpha}$ is unambiguously non-negative and $\dot{L}_Y$ is non-positive during transition. So if $\left\{\frac{1}{L_Y} - \frac{1}{1-L_Y}\right\} \le 0$, then this is sufficient (though not necessary) for *LSH* to be decreasing. Knowing that $0 \le L_Y \le 1$, we can numerical calculate that,

$$L_Y \ge 0.5 \qquad \rightarrow \qquad \text{then} \quad \left\{\frac{1}{L_Y} - \frac{1}{1-L_Y}\right\} \le 0.$$

So $L_Y > 0.5$ is a sufficient condition: if the *Y* sector employs at least half of the labor supply and the transition path of *LSH* to its steady-state is monotonic, then *LSH* converges to its steady-state value from above.

Now we consider the case of $L_Y < 0.5$. From equation (C.5) it follows that

$$\frac{\dot{\alpha}}{1-\alpha} + \left\{\frac{1}{L_Y} - \frac{1}{1-L_Y}\right\}\dot{L}_Y \ge 0$$

is a sufficient condition for LISH to be decreasing. From equation (4.2) in the main text we know that

$$\dot{\alpha} = K(1-\alpha)^2\left\{\left[log\left(\frac{K}{AL_Y}\right) + 1\right]\left(\frac{\dot{K}}{K}\right) - \frac{\dot{L}_Y}{L_Y}\right\} \text{ so } \frac{\dot{\alpha}}{(1-\alpha)} = K(1-\alpha)\left\{\left[\log\left(\frac{K}{AL_Y}\right) + 1\right]\left(\frac{\dot{K}}{K}\right) - \frac{\dot{L}_Y}{L_Y}\right\}$$

Therefore, the condition can be written as follows

(C.6) $$K(1-\alpha)\left\{\left[\log\left(\frac{K}{AL_Y}\right)+1\right]\left(\frac{\dot{K}}{K}\right)\right\}+\left\{\frac{1}{L_Y}-\frac{K(1-\alpha)}{L_Y}-\frac{1}{1-L_Y}\right\}\dot{L}_Y \geq 0$$

The first term of equation (C.6) is positive whenever savings are positive, so in order to prove that condition (C.6) holds, it suffices to prove that $\left\{\frac{1}{L_Y}-\frac{K(1-\alpha)}{L_Y}-\frac{1}{1-L_Y}<0\right\}$ or

$1-\frac{L_Y}{1-L_Y}<(1-\alpha)K$.

If $L_Y < 0.5$ it follows that $1-\frac{L_Y}{1-L_Y}<1$. So, given $L_Y < 0.5$, $K(1-\alpha)>1$ is a sufficient condition. Using equations (3.6) and (4.2) in the main text we get,

(C.7) $$\frac{\dot{L}_Y}{L_Y}=\left(\frac{I}{K}\right)\left\{\left[1+\log\left(\frac{K}{AL_Y}\right)\right]\xi-\left(\frac{1-\xi}{1-\alpha}\right)\right\},$$

and since $\frac{\dot{L}_Y}{L_Y}\leq 0$ we know that $(1-\alpha)\left[1+\log\left(\frac{K}{AL_Y}\right)\right]<\left(\frac{1-\xi}{\xi}\right)$.

Similarly form equation (3.5) we know that,

(C.8) $$\frac{\dot{K}}{K}=\xi\left(\frac{I}{K}\right)$$

Using (C.7) and (C.8), equation (C.6) can be written as,

$$\xi\frac{I}{K}\left[\left\{K(1-\alpha)\left[log\left(\frac{K}{AL_Y}\right)+1\right]\right\}+\left\{1-K(1-\alpha)-\frac{L_Y}{1-L_Y}\right\}\left(\frac{1}{(1-\alpha)}\right)\left\{(1-\alpha)\left[1+log\left(\frac{K}{AL_Y}\right)\right]-\left(\frac{1-\xi}{\xi}\right)\right\}\right]$$
≥ 0.

Rearranging,

(C.9)

$$\xi\frac{I}{K}\Bigg[\frac{1}{(1-\alpha)}\Bigg\{1-\frac{L_Y}{1-L_Y}\Bigg\}\Bigg\{(1-\alpha)\Bigg[1+log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]-\Bigg(\frac{1-\xi}{\xi}\Bigg)\Bigg\}+K\Bigg\{\Bigg(\frac{1-\xi}{\xi}\Bigg)-\alpha\Bigg[1+log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\Bigg]\Bigg\}\Bigg]$$

$\geq 0.$

If L < 0.5 then $1-\frac{L_Y}{1-L_Y}>0$ and the following inequality is a sufficient condition:

$$K\Bigg\{\Bigg(\frac{1-\xi}{\xi}\Bigg)-\alpha\Bigg[1+\log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\Bigg\}>\frac{1}{(1-\alpha)}\Bigg\{1-\frac{L_Y}{1-L_Y}\Bigg\}\Bigg\{\Bigg(\frac{1-\xi}{\xi}\Bigg)-(1-\alpha)\Bigg[1+\log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\Bigg\}.$$

Rearranging:

$$(1-\alpha)\Bigg\{K+\frac{\xi}{1-\xi}\Bigg(\Bigg\{1-\frac{L_Y}{1-L_Y}-\alpha K\Bigg\}\Bigg)\Bigg[1+\log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\Bigg]\Bigg\}>\Bigg\{1-\frac{L_Y}{1-L_Y}\Bigg\}.$$

We proved that $\xi\geq\alpha\rightarrow 1-\xi<1-\alpha$, therefore if

$$(1-\alpha)\Bigg\{K+\frac{\xi}{1-\xi}\Bigg(1-\frac{L_Y}{1-L_Y}-\alpha K\Bigg)\Bigg[1+\log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\Bigg]\Bigg\}>\Bigg\{(1-\alpha)K+\xi\Bigg\{1-\frac{L_Y}{1-L_Y}-\alpha K\Bigg\}\Bigg\}\Bigg[1+\log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]$$

then condition (C.9) holds.

So it suffices to prove that

(C.10) $$\Bigg\{(1-\alpha)K+\xi\Bigg\{1-\frac{L_Y}{1-L_Y}-\alpha K\Bigg\}\Bigg\}\Bigg[1+\log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\geq\Bigg\{1-\frac{L_Y}{1-L_Y}\Bigg\}.$$

First, if $1-\frac{L_Y}{1-L_Y}-\alpha K<0$ then $K>\frac{1}{\alpha}\Bigg(1-\frac{L_Y}{1-L_Y}\Bigg)$ and

$(1-\alpha)K>\frac{1-\alpha}{\alpha}\Bigg(1-\frac{L_Y}{1-L_Y}\Bigg)$. Now, if $(1-\alpha)K\geq\Bigg(1-\frac{L_Y}{1-L_Y}\Bigg)$, then

$(1-\alpha)K\Bigg[1+log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\geq\Bigg(1-\frac{L_Y}{1-L_Y}\Bigg)$ because $\Bigg[1+log\Bigg(\frac{K}{AL_Y}\Bigg)\Bigg]\geq 1$. Therefore,

whenever $\alpha\geq 0.5$ then $K(1-\alpha)>1$holds.

Second, if $1-\frac{L_Y}{1-L_Y}-\alpha K<0$ and $\alpha < 0.5$ then $K>2\left[1-\frac{L_Y}{1-L_Y}\right]$ and

$K>\left[1-\frac{L_Y}{1-L_Y}\right]$ so $K(1-\alpha)>1$ holds.

Third, if $1-\frac{L_Y}{1-L_Y}-\alpha K\geq 0$ and $\xi \geq \frac{1}{1+\log\left(\frac{K}{AL_Y}\right)}\frac{1-2L_Y}{1-2L_Y-\alpha K(1-L_Y)}$ then

condition (C.10) holds.

Fourth, $1-\frac{L_Y}{1-L_Y}-\alpha K\geq 0$ and $\xi < \frac{1}{1+\log\left(\frac{K}{AL_Y}\right)}\frac{1-2L_Y}{1-2L_Y-\alpha K(1-L_Y)}$ then

condition (C.10) holds whenever

$$(1-\alpha)K\left[1+\log\left(\frac{K}{AL_Y}\right)\right]\geq\left\{1-\frac{L_Y}{1-L_Y}\right\}\left(1-\xi\left[1+\log\left(\frac{K}{AL_Y}\right)\right]\right)+\alpha K\xi\left[1+\log\left(\frac{K}{AL_Y}\right)\right]$$

$1-\frac{L}{1-L}-\alpha K\geq 0$ so $\frac{1}{K}\left(1-\frac{L}{1-L}\right)\geq\alpha$ and it suffices to prove that

$$(1-\alpha)K\left[1+\log\left(\frac{K}{AL_Y}\right)\right]\geq\left\{1-\frac{L_Y}{1-L_Y}\right\}. \tag{C.11}$$

Since $1-\alpha=\frac{1}{1+K\log\left(\frac{K}{AL_Y}\right)}$ we can rewrite (C.11) as follows:

$$\frac{K+\log\left(\frac{K}{AL_Y}\right)}{1+K\log\left(\frac{K}{AL_Y}\right)}\geq\left\{1-\frac{L_Y}{1-L_Y}\right\}.$$

So a sufficient condition is $K\geq 1$.

Appendix D: Claim: ***There exists a*** $\tilde{K}$ ***such that*** $\forall\ K > \tilde{K}$ **,** $\left(\frac{K}{AL_{yt}}\right) < \left(\frac{1}{\rho}\right)^{\frac{1}{1-\alpha}}$ **.**

Define $\tilde{K} = \frac{1}{\log\left(\frac{1}{\rho}\right)}$. Since $\rho > 0$ then $\log\left(\frac{1}{\rho}\right)$ is a finite number and so is $\tilde{K}$.

Note that $\tilde{K} \geq \frac{\alpha}{\log\left(\frac{1}{\rho}\right)}$ because $\alpha \leq 1$. Therefore, $log\left(\frac{1}{\rho}\right) \geq \frac{\alpha}{\tilde{K}}$ and,

(D.1) $$\frac{1}{1-\alpha}\log\left(\frac{1}{\rho}\right) \geq \frac{1}{1-\alpha}\frac{\alpha}{\tilde{K}}$$

Combining equation (4.1) from the main text and (D.1) yields,

$$\frac{1}{1-\alpha}\log\left(\frac{1}{\rho}\right) \geq K\ \log\left(\frac{K}{AL}\right)\frac{1}{\tilde{K}}$$

Rearranging:

(D.2) $$\log\left(\frac{1}{\rho}\right)\frac{\tilde{K}}{K} \geq (1-\alpha)\log\left(\frac{K}{AL}\right)$$

Since we are considering $K > \tilde{K}$, $\log\left(\frac{1}{\rho}\right) > (1-\alpha)\log\left(\frac{K}{AL}\right)$ and $\left(\frac{1}{\rho}\right) > \left(\frac{K}{AL}\right)^{(1-\alpha)}$.

Therefore $\left(\frac{1}{\rho}\right) > \left(\frac{K}{AL}\right)^{(1-\alpha)}$ holds for any $K > \tilde{K}$.

Appendix E: Claim: $\left(\frac{K}{AL_Y}\right)^{\alpha-1}$ *increases as K increases.*

Consider a function,

(E.1)
$$G(K,L) = (\alpha - 1)log\left(\frac{K}{AL_Y}\right).$$

Differentiating with respect to time:

$$\frac{\partial G(K,L)}{\partial t} = \dot{\alpha}\log\left(\frac{K}{AL}\right) - (1-\alpha)\left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right).$$

The above can be combined with (4.2) from the main text to yield,

(E.3)

$$\frac{\partial G(K,L)}{\partial t} =$$

$$(1-\alpha)\left(K(1-\alpha)\left[log\left(\frac{K}{AL_Y}\right)\frac{\dot{K}}{K}\right]log\left(\frac{K}{AL}\right) - \left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right)\left(1 - K(1-\alpha)log\left(\frac{K}{AL}\right)\right)\right).$$

Using equation (4.1) and rearranging:

(E.4)
$$\frac{\partial G(K,L)}{\partial t} = (1-\alpha)^2\left(\left(\log\left(\frac{K}{AL_Y}\right)\right)^2 \dot{K} - \left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right)\right).$$

Therefore if $\dot{K}\left(\left(\log\left(\frac{K}{AL_Y}\right)\right)^2 - \frac{1}{K}\right) > -\frac{\dot{L}}{L}$ then $\frac{\partial G(K,L)}{\partial t} > 0$.

Using (A.23) and (A.5) we can rewrite the above condition as,

(E.5)
$$\xi\left(\left(log\left(\frac{K}{AL_Y}\right)\right)^2 - \frac{1}{K}\right) > -\left\{\left(\left(\frac{\alpha}{1-\alpha}\right)\frac{1}{K} + 1\right)\xi - \frac{(1-\xi)}{(1-\alpha)}\right\}.$$

Rearranging,

(E.6)
$$\frac{(1-\alpha)}{\alpha}\left(\alpha\left(log\left(\frac{K}{AL_Y}\right)\right)^2 + \alpha\, log\left(\frac{K}{AL_Y}\right) + \alpha - \frac{\alpha}{K}\right) > \frac{(1-\xi)}{\xi}.$$

We already know that $\alpha \le \xi$ so it remains to prove that,

$$\text{(E.7)} \qquad \alpha\left(log\left(\frac{K}{AL_Y}\right)\right)^2 + \alpha log\left(\frac{K}{AL_Y}\right) + \alpha - \frac{\alpha}{K} > 1 .$$

The function $\alpha log\left(\frac{K}{AL_Y}\right)$ is strictly increasing in *K* and $\lim_{K\to\infty} \alpha_t \, log\left(\frac{K}{AL_Y}\right) = \infty$.

Therefore, there exists a capital stock $\overline{K}$ such that $\alpha \log\left(\frac{K}{AL_Y}\right) > 1$ for any $K > \overline{K}$.

Consequently, if $K > \overline{K}$ then $log\left(\frac{K}{AL_Y}\right)\left(\alpha log\left(\frac{K}{AL_Y}\right) - 1\right) > 1 - \alpha$. rearranging it yields,

$$\text{(E.8)} \qquad \alpha\left(log\left(\frac{K}{AL_Y}\right)\right)^2 + \alpha log\left(\frac{K}{AL_Y}\right) + \alpha - \frac{\alpha}{K} > 1 .$$

Appendix F: Claim: ***There exists a*** K^{**} ***such that for any initial*** $K_0 > K^{**}$, $\frac{\dot{C}}{C} > 0$.

Define $K^{**} = \max(\tilde{K}, \overline{K})$. From the previous two propositions it follows that for any initial capital stock $K_0 > K^{**}$ the optimal growth rate of consumption is positive.